A Fast and Scalable Algorithm for Calculating the Achievable Capacity of a Wireless Mesh Network

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***Abstract*—This paper considers the problem of rapidly determining the maximum achievable capacity of a multi-hop wireless mesh network subject to interference constraints. Being able to quickly determine the maximum supported flow in a wireless network has numerous practical applications for network planners and researchers. Current approaches for determining network capacity either provide asymptotic results that are not necessarily achievable, are computationally intractable and cannot be computed quickly, or are not generalizable to different interference constraints for emerging technologies. In this paper, we present a new algorithm to rapidly determine the maximum concurrent flow for an arbitrary number of unicast and multicast connections subject to arbitrary binary interference constraints, and provide a feasible route and schedule to support those flows. The solution provided by our algorithm is within $O(\delta)$ of the optimal maximum flow, where δ is the maximum number of links that cannot be activated due to interference from some particular transmission. We use our algorithm to perform a network capacity analysis for emerging wireless technologies. We compare the achievable capacity of omni-directional, single-beam and multi-beam directional networks operating at different frequencies.**

I. Introduction

As the number of wireless devices continues to grow, new wireless technologies and protocols continue to be developed to interconnect these devices. A key metric in assessing the performance of these new technologies and protocols is to compare their performance with respect to the the maximum throughput that a network can support. While significant work has gone into understanding wireless network scalability [1], developing cross-layer optimization schemes for near-optimal network throughput [2], or determining bounds on network capacity [3], there still does not exist an approach that can rapidly determine the achievable capacity of a wireless network under interference constraints within a fixed bound of the optimal solution. Such a tool would have numerous applications for a network planner or researcher. Examples include determining the instantaneous capacity of a wireless network with known mobility patterns at any given point in time, quickly assessing the maximum achievable flow of a large number of potential network topologies or deployments, comparing achievable flows for different wireless technologies and their respective interference patterns against one another, and understanding how well wireless protocols perform versus the maximum achievable capacity.

In this paper, we present a new algorithm that can rapidly determines the maximum achievable concurrent flow for any number of unicast and multicast connections within a wireless network subject to wireless interference constraints. We label our approach the Fast Algorithm for Determining Wireless Network Capacity (FAD-WNC). In particular, we consider networks with known parameters, such as node placement, transmission pattern and distance, and link capacity. This is in contrast to asymptotic analysis that has been previously used to characterize scalability of network capacity for almost arbitrary network parameters [1], and does not easily allow insight into the behavior of actual deployed networks.

The joint routing and scheduling problem for a given set of demands is an NP-Hard cross-layer optimization problem [4], and just finding an optimal wireless transmission schedule for a predetermined set of links without the addition of routing is NP-Hard [5]. We effectively bypass the cross-layer optimization problem by leveraging the large body of existing work for rapidly determining the maximum concurrent flow in wired (no-interference) networks. The solution for a maximum flow in wired networks provides insight into how to schedule links to support a maximum wireless (with-interference) flow, for which we develop a fast approach to jointly schedule all demands. Our algorithm is able to determine a set of routes and a corresponding link activation schedule that achieves a network throughput that is within $O(\delta)$ of the network capacity upper bound, where δ is the maximum number of links that cannot be activated due to interference from some particular transmission. Additionally, we discuss approaches to dramatically increase the speed of the proposed algorithm with only limited decrease in the solution quality.

One of the primary motivations of our work is to have a tool that enables a network planner to quickly compare and contrast the achievable network capacity of emerging wireless technologies. Traditional wireless communications have used omni-directional antennas, where a user's transmission interferes with others users in all directions. Different papers have looked at finding feasible routes and schedules for networks of omni-directional users [6]. More recently, networks using directional antennas have been studied [7], where beams can be formed to or from specific users. Since transmissions can be

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targeted, directional networking allows for significantly lower interference over omni-directional systems, which results in higher network capacity.

New technologies such as smart-antennas with adaptive digital beamforming allow a user to selectively communicate simultaneously with multiple other users by forming either multiple transmit or receive beams, while also steering nulls towards interfering users [8]. This adaptive multi-beam communication system allows for almost arbitrary interference patterns between users depending on which set of beams and nulls are activated at any given moment in time. Our algorithm allows for rapidly determining the achievable network capacity under arbitrary interference constraints and allows for a common mechanism to examine the effect of different wireless technologies and their respective interference patterns on network capacity.

In this paper, we present the following novel contributions. The Fast Algorithm for Determining Wireless Network Capacity (FAD-WNC) that finds an achievable maximum network capacity of a wireless network subject to interference constraints. FAD-WNC runs in polynomial time and finds a feasible maximum flow that is within a guaranteed bound of the optimal solution. We use our algorithm to do a comparison of maximum network capacity using different wireless technologies operating at different frequencies. In particular, we use FAD-WNC to compare the capacity of omni-directional networks operating at 2.4 GHz to directional beamforming networks operating at the 24 GHz ISM bands. We examine single-beam directional antennas, as well as antennas capable of using multiple simultaneous transmit or receive beams. Depending on the choice of antenna and beamforming algorithms, different beamforming approaches have different gain and beamwidth and these different approaches come with different cost and complexity. We characterize the achievable network capacity to show when a particular technology has more benefit.

The paper is organized as follows. In Section II, we review of related work in this area. In Section III, we discuss the network model that we use for the rest of the paper. In Section IV, we present the Fast Algorithm for Determining Wireless Network Capacity (FAD-WNC) algorithm for quickly finding the achievable capacity of a wireless network. In Section V, we present simulation results of FAD-WNC, where we examine runtimes and compare the achievable network capacity for different wireless technologies and interference patterns.

II. Related Work

A large number of papers have examined the asymptotic behavior of networks. In particular, these works consider network capacity as the number of users goes to infinity. Typically transmission ranges, node placements, and other network parameters are configurable; finding feasible transmission schemes that achieve any capacity bound is usually not addressed. For omni-directional networks without mobility, [1] show that for n users, total network capacity scales at best $O(\sqrt{n})$. In other words, as the number of users in the network grows to infinity, the capacity available for any particular flow between users goes to zero. The analysis from [1] has been extended to different types of networks and interference patterns, including mobile networks [9], directional networks [10, 11], and MIMO relay networks [12]. While asymptotic results are useful at understanding fundamental network limits, they often do not say anything about capacities of finite node networks, nor how to actually achieve those capacity bounds. There have been efforts to characterize scalability and performance bounds of networks with more realistic parameters, with one example being [13], but these results still do not give feasible solutions.

For cross-layer optimization, numerous papers have been written that attempt to maximize some network metric (typically capacity) by finding a feasible solution consisting of a set of interference-free schedules and routes for the set of connections. Omni-directional networks are examined in [14–17], with each of these papers providing solutions that are within a guaranteed bound of optimal. Heuristic algorithms for routing and scheduling in directional networks are proposed in [18–20], but none provide any guaranteed bounds. For general interference constraints, [2, 5, 21] provide optimal solutions, but their proposed algorithms are computationally intractable and do not run in any guaranteed amount of time.

More recently, [3] finds an upper bound on wireless network capacity by maximizing the multi-commodity flow over the sparsest "wireless cut", which is an extension from the NP-Hard sparsest multi-commodity cut problem for wired networks. They demonstrate that the upper bound found by their algorithm is close to the optimal solution proposed by [21]. The upper bound though does not give feasible routes or transmission schedules for the demands in the network.

To the best of our knowledge, there has been no work that tries to find the maximum achievable capacity of networks using multi-beam adaptive antennas. There also does not appear to be any work that compares the maximum achievable network capacity in finite node networks of omni-directional and directional (both single beam and multi-beam) antennas.

III. Model and Problem Description

In this paper, we study the problem of how to quickly estimate the maximum achievable network capacity of a multi-hop wireless network subject to arbitrary binary interference constraints. Given a traffic demand, our algorithm finds a set of routes with corresponding interference-free transmission schedules for all demands in the network such that the minimum rate of any particular demand is maximized. Without interference-constraints, this has been traditionally known as the Maximum Concurrent Flow Problem [22]. We consider both unicast and multicast flows. For the case of a network with both unicast and multicast flows, polynomial-time algorithms are developed that achieve a maximum concurrent flow that is $O(\delta)$ from the optimal solution, where δ is the maximum number of links that cannot be activated due to interference from some particular transmission. We leverage the large body of work developed to rapidly determine the maximum concurrent flow in wired networks for both unicast [22] and multicast [23] traffic to develop rapid algorithms for calculating network capacity in wireless networks.

The network model is as follows. We are given a graph G with a set of wireless nodes V and links E. Link (i,j) has capacity $c_{ij}, \forall(i,j) \in E$. The number of nodes in the network is $n = |V|$. We assume all link capacities are rational values. The antenna patterns of each user is assumed to be known, which allows us to formulate interference constraints accordingly. We consider a stationary network; if a network is mobile, we consider a stationary snapshot of the network. Hence, we assume that the wireless nodes are static, and that the set of edges E is fixed. The number of neighbors node v has is called its *degree*, and the maximum degree in graph G is labeled as $\Delta(G)$.

We assume that the network uses a synchronous time slotted system with equal length time slots with a repeating schedule of T time slots.

We are given a set of unicast and multicast demands. For a unicast flow, d_{ij} of flow must be sent from user i to user j. For a multicast flow, d_{sS} of flow must be sent from source s to all of the multicast members S. We note that if $|S| = 1$, the multicast flow is a unicast flow. The goal of the maximum concurrent flow problem is to maximize the minimum fraction of any demand that can be achieved subject to all constraints on the network. In wired networks, the primary constraint is that no flow on a link can be greater than that link's capacity. In wireless networks, interference constraints are added, and certain links cannot be activated simultaneously or else their transmissions will interfere with one another.

In this paper, we use the binary interference model. For any pair of links, (i,j) and (k,l), either both links can be active simultaneously if they do not interfere with one another, or at most one of these links can be active if they do interfere [24]. An interference matrix $\mathcal{I}$ is defined where $I_{ij}^{kl} \in \mathcal{I}$ is 1 if links (i,j) and (k,l) can be activated simultaneously (do not interfere with each other), and 0 otherwise. Similar to [25], we define F_{ij} as the *interference region* of link (i,j). F_{ij} is the set of other links that cannot be active simultaneously with (i,j). We define δ to be the maximum-sized interference region: $\delta = \max_{(i,j)\in E} |F_{ij}|$.

Binary interference is used for the K-hop interference model [26], where if link (k,l) is less than K hops away from link (i,j), the two links will interfere. The 1-hop interference model has been used to represent interference in directional networks [27], and the 2-hop interference model has been used to represent omni-directional interference [26]. More generally, the binary interference model can be used to represent almost arbitrary interference constraints. In directional networks, any realistic transmission pattern will have some beamwidth associated with it, which will cause a transmission to interfere with a set of users. The binary interference model can capture the interference from various beamwidths and from activating any particular subset of beams from a multi-beam antenna.

In our analysis, we use a *conflict graph* [5], which is used to represent the interference in a network using a binary interference model. We construct a conflict graph $\bar{G}^c$ as follows: a node v_{ij} is added for each link (i,j) in the transmission graph G, and a link is added between two nodes v_{ij} and v_{kl} in $\bar{G}^c$ if links (i,j) and (k,l) interfere with one another in G. We note that the neighbors of node v_{ij} in $\bar{G}^c$ are the nodes associated with the interference region F_{ij} for link (i,j). Any independent set[1] of $\bar{G}^c$ are a set of edges in the transmission graph G that can be activated simultaneously without interference. The maximum degree of the conflict graph, $\Delta(\bar{G}^c)$, is the maximum number of links that cannot be activated due to interference from some particular transmission. Hence, $\delta = \Delta(\bar{G}^c) = \max_{(i,j)\in E} |F_{ij}|$.

IV. Algorithm for Rapidly Determining Wireless Network Capacity

In this section, we present our algorithm for quickly determining wireless network capacity. This problem is a cross-layer optimization, and as discussed in Section II, existing solutions for jointly determining the set of routes and schedules for arbitrary interference constraints have been computationally intractable. The routing portion of such a joint optimization problem selects the optimal set of links to traverse such that interference is minimized. Depending on how many demands traverse any given link, certain links in an optimal solution will be more heavily utilized, and as such these links require more time slots than lightly utilized links.

Instead of trying to jointly optimize, we leverage the large body of existing work for rapidly determining maximum concurrent flow in wired (no-interference) networks. We wish to understand which links are more heavily utilized for a maximum flow, allowing us to assign those links a commensurate number of time slots. The maximum concurrent flow for wired networks gives us a rapid method of determining the utilization of any link for some maximum flow. We note that these link utilizations are not necessarily optimal for a *wireless* maximum concurrent flow, but they offer insight into how some sort of maximum flow can be allocated. We next develop a scheduling approach that assigns time slots to the various links in the network such that at least $O(\delta)$ of the network capacity upper bound can be supported. We call the set of routes and link utilizations for the wired maximum concurrent flow the No-Interference Maximum Flow (NI-MF), and we call the wireless maximum flow the With-Interference Maximum Flow (WI-MF).

The maximum concurrent flow for networks without interference for both unicast and multicast flows is well studied; hence, the focus of this paper is presenting an algorithm for quickly determining an interference-free schedule for the maximum concurrent flow in wireless networks subject to interference constraints. In Section IV-A, we give a brief overview of the various methods of solving for NI-MF for both unicast and multicast flows. In Section IV-B, we present our algorithm to find an interference-free schedule. In Section IV-C, we evaluate the performance of FAD-WNC.

[1] An independent set is a set of nodes where no two nodes are the end points of the same edge.

A. *Finding the Maximum Concurrent Flow for a Network without Interference Constraints*

In this section we discuss different approaches for finding the No-Interference Maximum Flow. The maximum concurrent flow problem seeks to maximize the minimum fraction of each connection that can be supported in a capacitated network. There is a large body of literature that we can leverage for computing the maximum concurrent flow for both the unicast and multicast case. A linear program that optimally solves for the maximum concurrent flow problem in polynomial time has been previously provided in [28] for the unicast case and in [23] for the multicast case.

While linear programs provide optimal solutions, they are not necessarily efficient to solve in practice. The authors of [23] benchmarked the performance of their linear programming formulation for the multicast maximum concurrent flow and found that runtime can easily take hours for moderately sized networks. In [22], a survey of approximation algorithms for the unicast maximum concurrent flow problem are presented, as well as a new algorithm that performs faster than previous versions. To calculate a $(1+\epsilon)$ approximation of the optimal unicast maximum concurrent flow, [22] develops an algorithm that runs in $O(\epsilon^{-2}(k+m)m)$ time, where k is the number of unicast connections and m is the number of edges in the network. For the multicast case, [23] provides an algorithm that achieves the optimal solution. Their approach does not have guaranteed polynomial runtime, but they demonstrate that in practice it rapidly achieves the optimal solution. Their simulation results show the multicast maximum concurrent flow can be solved in under one second for 1000 node networks. We note that their approach can be used for the unicast case as well: a multicast session from source s to a single destination d is identical to a unicast flow between s and d.

B. *Finding an Interference-Free Schedule*

In this section, we assume that the NI-MF has already been found using one of the approaches discussed above, and we seek to find the WI-MF. The objective of our problem is to find an interference-free schedule that maximizes individual link rates, while supporting at least $O(\delta)$ of the wired network capacity.

We assume that NI-MF provides a feasible flow decomposition for the wired maximum network capacity; i.e., flow values f_{ij} for each link $(i,j) \in E$. We define the *no-interference link utilization* of link (i,j) as:

$$u_{ij} = \frac{f_{ij}}{c_{ij}} \tag{1}$$

If a final schedule has a total of T time slots and some link (i,j) is assigned t_{ij} time slots, then link (i,j) is active for a fraction $\frac{t_{ij}}{T}$ of time and supports a total flow of $\frac{t_{ij}}{T}c_{ij}$. We define the *with-interference link utilization* for link (i,j) as:

$$w_{ij} = \frac{t_{ij}}{T} \tag{2}$$

If $w_{ij} \geq u_{ij}$ (i.e., the with-interference link utilization is greater than the no-interference link utilization), then flow f_{ij} can be fully supported on link (i,j) from the initial NI-MF. However, if $w_{ij} < u_{ij}$, then only a fraction of f_{ij} can be supported on the link. The ratio of without-interference to with-interference flow supported on link (i,j) is:

$$\sigma_{ij} = \frac{w_{ij}}{u_{ij}} \tag{3}$$

While the flow decompositions that NI-MF provides may not be optimal for the wireless maximum flow, any schedule that maintains $\sigma_{ij} \geq \sigma$, $\forall (i,j) \in E$ will be within $O(\sigma)$ of NI-MF for any constant σ. We know any maximum flow solution with interference can never be greater than the solution without interference. Our algorithm finds a feasible flow; hence, it is a lower bound on any optimal solution. For our solution, by guaranteeing that $\sigma_{ij} > \delta^{-1}, \forall (i,j) \in E$, we can guarantee that the achieved maximum concurrent flow in the wireless network with interference is at most $O(\delta)$ from optimal.

First, we present a high-level outline of the scheduling algorithm. Afterwards, each step is discussed in more detail, and potential speed improvements are explored.

1) Algorithm Outline: The steps of the scheduling algorithm are the following.

(i) **Convert to integer:** Convert all no-interference link utilizations u_{ij} to integer values z_{ij} by multiplying by some integer R.

(ii) **Create multi-edge conflict graph:** We create a *multi-edge* conflict graph, which is a variant of the traditional conflict graph. Link (i,j) will be represented by z_{ij} nodes that form a clique[2]; we label this conflict graph $\bar{G}^M$. If links (i,j) and (k,l) cannot be active simultaneously in G, then in $\bar{G}^M$, each node from clique (i,j) will have a connection to each node in clique (k,l).

(iii) **Color conflict graph:** Find a minimum graph coloring of $\bar{G}^M$, where each color represents a time slot of the final schedule. The total number of colors is the number of time slots for a schedule. Since the graph coloring problem is strongly NP-Complete [29], we use the Welsh-Powell greedy coloring algorithm that has a runtime of $O(n\Delta(\bar{G}^M))$ and finds a solution that uses at most $\Delta(\bar{G}^M)+1$ colors [30].

(iv) **Scale flow:** Define $\sigma_{min} = \min_{\forall (i,j)\in E} \sigma_{ij}$. After scheduling, each link can support at least σ_{min} of the No-Interference Maximum Flow. Scale the maximum concurrent flow for the network without interference by σ_{min} to find an achievable maximum flow in a network with interference. We will demonstrate that $\sigma_{min} \geq (\delta+1)^{-1}$.

2) Algorithm Discussion: We now discuss each step of the algorithm in more detail. The key is to assign time slots to the different links such that the achieved flow on any link can support at least $O(\delta)$ of the initial flow. Hence, our goal is find an interference-free schedule such that: $w_{ij} \geq \frac{1}{\delta+1}u_{ij},\ \forall (i,j) \in E$.

In step (i), the no-interference link utilizations u_{ij} are converted to integer values z_{ij}, $\forall (i,j) \in E$. To do so, we find

[2]A clique are a set of nodes that are all connected to one another.

some integer value R such that $R \cdot u_{ij} \in \mathbb{Z}$, $\forall(i,j) \in E$, where $\mathbb{Z}$ is the set of integers. We demonstrate in Lemma 1 that all link utilizations u_{ij}, $\forall(i,j) \in E$ are rational values.

The runtime of the coloring algorithm for some graph is $O(n\Delta)$, where n is the number of nodes in that graph and Δ is the maximum degree of that graph. Since the number of nodes in the multi-edge conflict graph scale with R and the runtime of the graph coloring algorithm is a function of the number of nodes in the graph, it is important that we demonstrate that R is polynomial bounded by the size of the input variables. We demonstrate this to be the case in Lemma 1.

Lemma 1. *All link utilizations u_{ij}, $\forall(i,j) \in E$ are rational values. Furthermore, there exists some integer R such that $R \cdot u_{ij} \in \mathbb{Z}$, $\forall(i,j) \in E$, where R is polynomial bounded by the size of the input variables.*

The proof for Lemma 1 is presented in Appendix A. We note that demonstrating the existence of an R that is polynomial-bounded in size with respect to the inputs is important from an analytic perspective, but not necessarily from a practical perspective. The value R is a scaling factor that determines the size of cliques for the multi-edge conflict graph. Since the runtime of any graph coloring algorithm is dependent on the number of nodes in the graph, having a large value of R can result in slower algorithm performance. Smaller values of R can be used by dropping any fractional values to improve runtime while still producing high-fidelity results. We consider these performance trade-offs in Section V.

Next, we discuss steps (ii) and (iii) of the algorithm. A typical conflict graph construction represents any individual link (i,j) in G as a single node v_{ij}; we call this conflict graph construction $\bar{G}^1$. In our solution approach, we construct a multi-edge conflict graph $\bar{G}^M$ where each link (i,j) in G is represented by z_{ij} nodes that form a clique. Since no two nodes of a clique can share the same color, a clique of size z will require exactly z colors. Recall that $z_{ij} = R \cdot u_{ij}$. A final graph coloring preserves the ratio of utilization factors across all links:

$$\frac{R \cdot u_{ij}}{R \cdot u_{kl}} = \frac{z_{ij}}{z_{kl}}, \ {}_{\forall(k,l)\in E}^{\forall(i,j)\in E}$$

If R is reduced in size and the fractional value is discarded, then each link receives fewer time slots. But, fewer time slots overall will be needed for a feasible schedule, and the ratio of link activations will be still roughly preserved; each link will still be active a similar fraction of time and hence, will be able to support a similar amount of flow.

In step (iv), we scale the initial maximum concurrent flow given by NI-MF such that it can be supported by the interference-free schedule that was found in step (iii). If the total number of colors (i.e., time slots) to color $\bar{G}^M$ is T and link (i,j) uses z_{ij} time slots, then link (i,j) has a with-interference link utilization of $w_{ij} = \frac{z_{ij}}{T}$. If $w_{ij} \geq u_{ij}$, then the full flow on link (i,j) can be supported, and if $w_{ij} < u_{ij}$, then only the fraction $\sigma_{ij} = \frac{w_{ij}}{u_{ij}}$ can be supported. Define $\sigma_{min} = \min_{\forall(i,j)\in E} \sigma_{ij}$. We can scale the initial solution given by NI-MF by σ_{min} to find an achievable maximum concurrent flow in the wireless network with interference constraints.

We now demonstrate that $\sigma_{min} \geq (\delta+1)^{-1}$, and hence our algorithm always produces a solution that is $O(\delta)$ of optimal. Recall that in conflict graph $\bar{G}^1$, each link (i,j) of G is represented by a single node v_{ij} in $\bar{G}^1$, and two nodes in $\bar{G}^1$ have a connection if and only if they cannot be activated simultaneously. Hence, the maximum degree of the conflict graph $\bar{G}^1$ is the maximum number of links that cannot be activated due to interference from some particular transmission, and $\delta = \Delta(\bar{G}^1)$.

Theorem 1. *The Fast Algorithm for Determining Wireless Network Capacity finds a feasible maximum concurrent flow for a network that is always within $O(\delta)$ of the optimal solution.*

Proof: NI-MF is the upper bound on any achievable solution for WI-MF. We will demonstrate that the supportable flow on every link in the WI-MF solution is within $\delta+1$ of the flow on that link in the NI-MF solution. Specifically, we will demonstrate:

$$\frac{w_{ij}}{u_{ij}} \geq \frac{1}{\delta+1}, \ \forall(i,j) \in E$$

We define the following two values: $z_{max} = \max_{\forall(i,j)\in E} z_{ij}$ and $u_{max} = \max_{\forall(i,j)\in E} u_{ij}$.

By using the Welsh-Powell coloring algorithm, conflict graph $\bar{G}^1$ can be colored using $\delta+1$ colors [30], where δ is the maximum degree of $\bar{G}^1$. In the multi-edge conflict graph $\bar{G}^M$, the i^{th} clique of size z_i will be colored using z_i colors. Hence, $\bar{G}^M$ can be colored with at most $T \leq (\delta+1)z_{max}$ colors.

To compute z_{ij} for any particular link, we multiplied each link utilization ratio u_{ij} by some value R: $z_{ij} = R \cdot u_{ij}$. By definition: $z_{max} = R \cdot u_{max}$.

The no-interference link utilization $u_{ij} = \frac{f_{ij}}{c_{ij}}$ is the total percentage of link capacity that is used to support the maximum concurrent flow in the network without interference. Maximum concurrent flow is achieved when the multi-commodity minimum-cut is saturated, and the minimum-cut is saturated when all of its respective links are allocated at capacity [31]. Hence, there exists some link (k,l) that is allocated at capacity; i.e., $f_{kl} = c_{kl}$, and $u_{kl} = \frac{f_{kl}}{c_{kl}} = 1$. A link can be utilized at most at 100%; therefore, $u_{max} = 1$, and $z_{max} = R \cdot u_{max} = R$.

Using $T \leq (\delta+1)z_{max}$, the with-interference link utilization w_{ij} has the following bound:

$$w_{ij} = \frac{z_{ij}}{T} \geq \frac{z_{ij}}{(\delta+1)z_{max}} = \frac{z_{ij}}{(\delta+1)R}, \ \forall(i,j) \in E$$

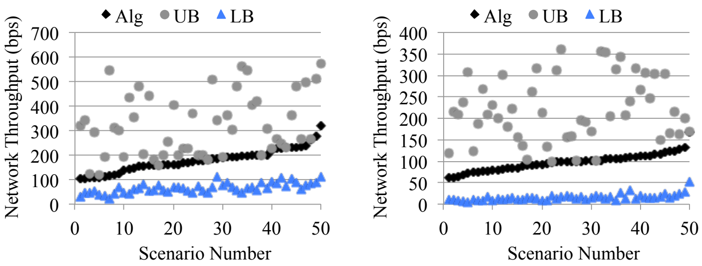
Using $u_{ij} = \frac{z_{ij}}{R}$, we complete the proof:

$$\frac{w_{ij}}{u_{ij}} \geq \frac{\frac{z_{ij}}{(\delta+1)R}}{\frac{z_{ij}}{R}} = \frac{1}{\delta+1}, \ \forall(i,j) \in E$$

■

C. Algorithm Performance Evaluation

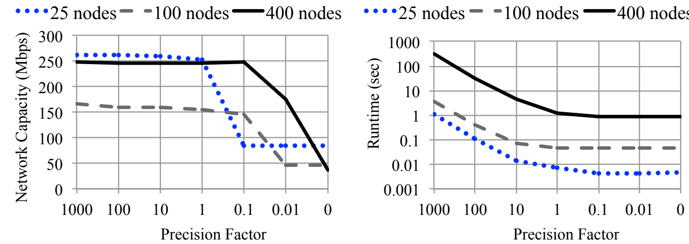
In this section, we evaluate the performance of our algorithm with respect to runtime and how far its achieved network capacity is from network capacity upper bound.



(a) 1-hop interference

(b) 2-hop interference

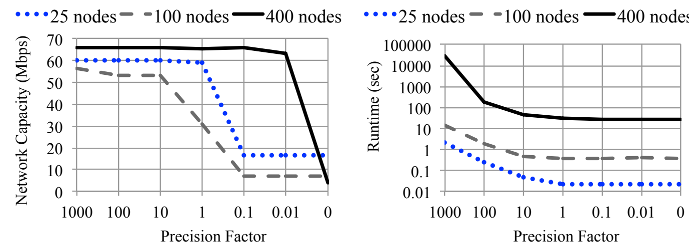
Fig. 1: Achieved wireless network capacity of our algorithm (Alg) compared to the guaranteed algorithm lower bound (LB) and the with-interference capacity upper bound (UB).



(a) Throughput with 1-hop interference

(b) Runtime with 1-hop interference

Fig. 2: 1-hop interference: capacity and runtime results.



(a) Throughput with 2-hop interference

(b) Runtime with 2-hop interference

Fig. 3: 2-hop interference: achieved throuhgput and runtime results.

We first compare the achieved wireless network throughput of our algorithm to both the algorithm's guaranteed lower bound and to the maximum with-interference capacity upper bound. For the upper bound capacity, we implement the maximum multi-commodity wireless cut algorithm proposed in [3]. We generate 50 random 100 node networks, and we consider uniform all-to-all unicast traffic. Since the algorithm in [3] was designed to work for networks with unit capacity links, we also consider networks of 1 Mbps links. We consider both 1-hop and 2-hop interference. In Fig. 1, each test result is plotted in ascending order of the algorithm's achieved network throughput. On average FAD-WNC achieves a network capacity that is close to the network capacity upper bound. For 1-hop and 2-hop interference, our algorithm finds a solution that is on average within a factor of 1.6 and 2.2 of the upper bound, respectively. In about 10% of instances, the algorithm achieved a solution that was within 10% of the upper bound network capacity. We also observe that our algorithm on average performs substantially better than its guaranteed lower bound. For 1-hop interference, the algorithm is 4.8 times better than the guaranteed lower bound, and for 2-hop interference, the algorithm is 13.7 times better.

We next wish to characterize performance trade-off for lower values of R with respect to runtime. In step (i) of the algorithm, we convert all no-interference link utilizations u_{ij} to integer values z_{ij} by multiplying by some integer R. As noted in the discussion, the runtime of the algorithm is dependent on R, and performance gains can be achieved by using smaller values of R and discarding any fractional components of z_{ij}. The value R can in fact be quite large, which will cause a significant slow-down in algorithm performance.

Label P the *precision factor*. To determine R, we consider the following approach. Let R be some power of 10 such that each link flow Ru_{ij} is greater than P. More precisely, let $R = 10^x$ for the minimum integer x such that $z_{ij} = \lfloor Ru_{ij} \rfloor \geq P$, $\forall (i,j) \in E$. We drop the fractional component of z_{ij}.

Smaller values of P (and hence, R) will provide less precision, but still generally preserve ratios of link utilization factors to one another. It is possible that $u_{ij} \gg u_{kl}$. If this is the case, then Ru_{ij} will be potentially be very large, and since a clique for link (i,j) in the multi-edge conflict graph will have size Ru_{ij}, the graph coloring algorithm may experience slow run-time. Hence, we allow P to be less than 1. For $P < 1$, there will exist at least one link (i,j) such that $0 < Rf_{ij} < 1$. For these links, set $z_{ij} = 1$. Hence, $z_{ij} = \max(\lfloor Rf_{ij} \rfloor, 1)$.

When the precision factor is 0, then every link in the network gets exactly one time slot; this results in the traditional conflict graph $\bar{G}^1$. A schedule for $\bar{G}^1$ requires at most $\Delta(\bar{G}^1)+1 = \delta+1$ time slots, and if every link is activated for one time slot, then each link can support $(\delta+1)^{-1}$ flow. This in fact is our worst-case capacity bound guarantee. Larger precision factors only improve the solution from the worst-cast guarantee.

We perform the following test to evaluate runtime. We generate 50 random graphs of 25, 100, and 400 nodes. Each graph has an average degree of 15. Capacities are randomly assigned to each link from a uniform distribution between 0 and 100 Mbps. For maximum concurrent flow, we consider uniform all-to-all unicast traffic; i.e., every node desires to send an identical amount of flow to every other node. To find the No-Interference Maximum Flow (NI-MF), we use the algorithm presented in [22]. We consider the 1-hop and 2-hop interference model: if link (k,l) is within K hops of link (i,j), the two links will interfere. We vary the precision factor P between 0.01 and 1000. We also consider $P = 0$, where the resulting schedule will assign one time slot per link. All algorithms were implemented in C, and all tests were run on a 2013 MacBook.

The runtimes presented are only for the scheduling algorithm, and not the NI-MF. The reason we did not include the runtime for NI-MF is that there are many approaches for doing so, and the focus of this paper is on finding a maximum flow *with* interference. In our implementation, for all test cases, the NI-MF algorithm (from [22]) ran in under 1 second.

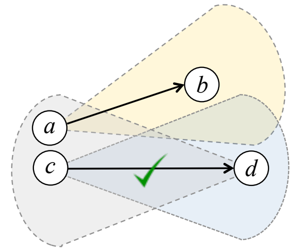
The results for 1-hop and 2-hop interference are plotted in Figs. 2 and 3, respectively. Both 1-hop and 2-hop interference have similar trends. There is little change in the achieved network capacity as the precision factor decreases from 1000

to 1. For 1-hop interference, 25, 100, and 400 node networks see a decrease in achieved network capacity of 3%, 6%, and 1%, respectively. For 2-hop interference, there is a decrease in achieved network capacity of 2%, 8%, and 1% for 25, 100, and 400 node networks, respectively. The drop in algorithm runtime associated with a reduction in the precision factor is significant. For both 1-hop and 2-hop interference, run time drops by over 99%. For the 400 node network with 1-hop interference, runtime goes from 327 seconds at $P = 1000$ to 1.2 seconds at $P = 1$. With 2-hop interference, 400 node network has a runtime decrease from 29,561 seconds at $P = 1000$ to 30 seconds at $P = 1$. The 2-hop interference model has significantly higher conflicts than the 1-hop interference case. This increased interference causes the graph coloring algorithm to have a longer search for an available color for each link, which causes the longer runtime.

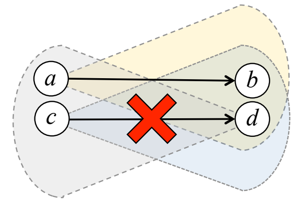
For precision factors less than 1, we observe that the achieved network capacity declines first for the 25 node network, then the 100 node network, and finally the 400 node network. For smaller networks with $P < 1$, after being multiplied by R, many links will end up with a single time slot in the final schedule. In larger networks, many links still have sufficiently high utilization such that they will still assigned a commensurate amount of time slots. The 400 node case only sees a significant drop when $P = 0$, and all links are assigned a single time slot. But, the savings in algorithm runtime do not seem to worth the drop in achieved network capacity. Most of the savings comes from $P = 1000$ to $P = 1$. Going from $P = 1$ to $P = 0$ sees at most a 4 second drop for the 400 node network with 2-hop interference, but sees an achieved network capacity drop of 13x. For all other cases, the time savings is below 0.5 seconds.

V. Network Capacity Evaluation for Different Wireless Technologies

In this section, we use our algorithm to compare network capacity for wireless technologies that experience different types of interference constraints. Since different technologies have different costs associated with them, the goal is to understand when a certain technology choice is appropriate and worth the cost. We evaluate network capacity as a function of beamwidth for directional antennas in the 24 GHz ISM band, and compare to omni-directional antennas in the 2.4 GHz ISM band. We consider sparse and dense networks in both terrestrial and airborne settings. For directional networks, we consider systems that can support either only a single transmit or receive beam, or can support multiple simultaneous transmit or receive beams [32]. We assume the use of directional antennas that are multi-element phased arrays, where gain, beamwidth, and interference patterns vary with the number of array elements and beamforming algorithms. Since multi-beam antennas are potentially much more expensive and complex, it is important to characterize when a multi-beam approach makes sense over a single-beam. Furthermore, we demonstrate that high-frequency directional systems do not always outperform low-frequency (and lower cost) omni-directional systems.



(a) Successful transmission



(b) Failed transmission

Fig. 4: Interference using the flat-top antenna model.

The 24 GHz ISM band has been considered for future 5G network design [33]. Atmospheric absorption loss at 24 GHz is around 0.1 dB/km [34], while at 2.4 GHz such loss is below 0.001 dB/km [34]. Thus, increased antenna gain is necessary to successfully operate in the 24 GHz band.

Antenna gain g is a function of the antenna type and the number of elements. We consider (isotropic) omni-directional antennas with a single antenna element that have a gain of $g = 0$ dBi. Phased antenna arrays are able to focus energy into directional beams to amplify a signal relative to an isotropic antenna, and can be used at both transmitting and receiving systems. For directional systems we use the values for antenna gain shown in Table I; these values either come from [35] or, for those marked with an asterisk (*), were derived using a similar methodology. We note that arrays with a large number of elements are being developed to support narrow beamwidth and high gain [36].

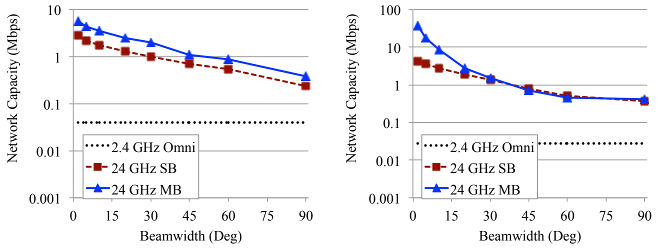
There are a variety of potential antenna patterns from different beamforming approaches. To simplify our analysis, we assume an ideal "flat-top" directional antenna model with constant gain across a beam that is θ degrees wide; any transmissions or receptions falling outside of this beam can be ignored [37]. All transmissions have some maximum distance, and interference constraints are as follows. A transmission from node c to node d can succeed only if there is no simultaneous transmission from node a such that both (i) d is within the transmit beam pointing from a to any node b and (ii) a is within the receive beam pointing from d to c, accounting for the maximum range of all beams. This example is illustrated in Fig. 4. We model interference omni-directional antennas as having a beamwidth of $\theta = 360°$.

All antennas considered are half-duplex; i.e., a node cannot both transmit and receive simultaneously. For multi-beam antennas, we add the following constraints for reception and transmission: (1) a node can receive simultaneous directional beams from users that are separated by at least $\frac{\theta}{2}$ degrees, and (2) a node cannot transmit simultaneously to multiple users that are spatially separated by less than $\frac{\theta}{2}$ degrees.

Our experiments consider random topologies with 50 nodes and a uniform all-to-all unicast traffic model. We consider both terrestrial and airborne networks. For terrestrial networks, trans-

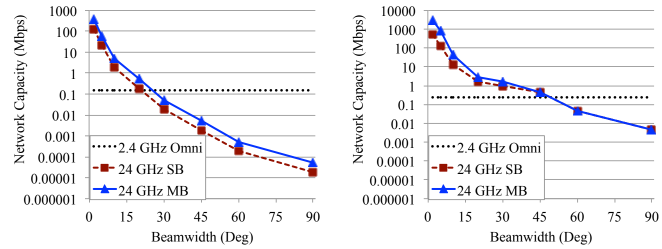
TABLE I: Directional beamwidth vs. gain.

Beamwidth θ	90°*	60°	40°	20°	10°	5°*	2°*
Gain g in dBi	5	10	14	20	26	32	38



(a) Sparse terrestrial network (b) Dense terrestrial network

Fig. 5: Terrestrial achievable network capacity vs. beamwidth for single-beam (SB) and multi-beam (MB) systems.



(a) Sparse airborne network (b) Dense airborne network

Fig. 6: Airborne achievable network capacity vs. beamwidth for single-beam (SB) and multi-beam (MB) systems.

missions have a maximum range of 1 km and transmit power is set to 1 Watt. For airborne networks, transmissions have a maximum range of 200 km and, to overcome these longer distances, transmit power is set to 10 Watts. For directional transmissions, the beamwidth (and the corresponding gain) is varied between 2° and 90°. We choose bandwidths of 10 MHz for the 2.4 GHz band and 100 MHz for the 24 GHz band, as spectrum is less scarce in the higher frequency range.

We test both sparse and dense networks, and consider 50 random graphs for each case. Nodes are spread further apart in sparse networks, thus node degrees are lower and paths will typically be longer. We observe an average node degree of 7 in our sparse networks and 30 in our dense networks. We assume free-space path loss, and that there are no blockages between users. Link capacities are set according to the Shannon capacity, which takes into account the antenna gain, bandwidth, atmospheric absorption, and free-space path loss between nodes. We assume a noise spectral density of $N_0 = -174$ dBm/Hz.

Results are shown in Figs. 5 and 6 for terrestrial and airborne networks, respectively. We first observe that in both terrestrial and airborne networks, narrower beamwidths result in significantly higher network capacity. This is due to the higher gain and lower interference associated with the narrower beamwidth. As beamwidth gets wider, the benefits of directionality decrease. In the airborne network, the reduced antenna gain associated with wider beamwidths cannot overcome the atmospheric absorption losses at 24 GHz for long-distance links. At a beamwidth of approximately 20° in the sparse airborne network and 45° in the dense airborne network, the achieved capacity of high-frequency directional network and low-frequency omni-directional network become the same. The reason the dense network can support wider beamwidths is that nodes are closer together on average than in the sparse network, and hence, users experience lower absorption loss and can support links with lower overall gain. For this exact reason (users being closer to one another), the short-range terrestrial network does not experience a "break-even" point between high-frequency directional networking and the low-frequency omni-directional system. Wink distances are small enough that atmospheric absorption does not result sufficient degradation of link quality, and wider beamwidth still provide substantial improvement over low-frequency omni-directional networks.

We next examine the performance of multi-beam (MB) directional networking versus the single-beam (SB) approach. For sparse networks, we observe the multi-beam approach achieves a network capacity that is approximately twice that over the single-beam approach for all beamwidths. For dense networks, we observe the multi-beam approach yields a network capacity that is approximately ten times that of the single-beam approach for narrow beamwidths, but this improvement disappears as beamwidth becomes wide. Here, the capacity improvement for narrow beams in dense networks is more significant due to the reduction in interference, allowing many transmit or receive beams to be simultaneously activate. Interference increases as beamwidth becomes wider, causing transmissions to overlap with unintended recipients. As can be seen, as beamwidth approaches 30° in a dense network, almost all of the benefit of a multi-beam system disappears. In sparse networks, nodes have fewer neighbors to simultaneously transmit to using narrow beams and, conversely, fewer neighbors to interfere with when using wider beams.

VI. Conclusion

In this paper, we considered the problem of rapidly determining the maximum achievable capacity of a multi-hop wireless mesh network subject to interference constraints. We present the Fast Algorithm to Determine Wireless Network Capacity (FAD-WNC) that quickly finds the maximum achievable network capacity for an arbitrary number of unicast and multicast connections subject to arbitrary binary interference constraints. The solution provided by FAD-WNC is within $O(\delta)$ of the optimal maximum flow, where δ is the maximum number of links that cannot be activated due to interference from some particular transmission. Our algorithms performs well with respect to optimal wireless maximum flow. We also suggest speed improvements for FAD-WNC, and demonstrate that our algorithm can find the maximum network flow in under 1 second for a 100 node network. We then use our algorithm to perform a network capacity analysis comparing different wireless technologies, including omni-directional, single-beam, and multi-beam directional antennas operating at different frequencies. We show that depending on the technology in use and the network characteristics, different approaches perform better than others, and that the more complex approach is not always better. We plan to use our tool to benchmark performance of different wireless protocols used in industry and academia, and to characterize the performance of different technologies.

Appendix

A. Proof for Lemma 1

As assumed by the network model, all inputs to our problem are rational, including all link capacities c_{ij}, $\forall(i,j) \in E$. As discussed in Section IV-A, NI-MF can be solved using a linear program (LP). The output of the LP are a set of flow allocations on each edge: f_{ij}, $\forall(i,j) \in E$. In an optimal solution given by an LP, the size of any output variable (i.e., the number of bits necessary to represent that variable) is polynomially bounded by the size of the inputs [38]. Since all of the inputs to our problem are rational (hence requiring a finite and bounded number of bits to represent), all output variables f_{ij}, $\forall(i,j) \in E$ are also rational and polynomial bounded by the size of the inputs. Since link capacities are rational, the no-interference link utilization $u_{ij} = \frac{f_{ij}}{c_{ij}}$ is also rational. Therefore, there exists some value R that is polynomially bounded by the size of inputs such that $R \cdot u_{ij} \in \mathbb{Z}$, $\forall(i,j) \in E$. $\square$

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